Spectral Gaps in the Dirichlet Problem for the Laplacian on a Cylinder in $\mathbb{R}^3$ Perforated by a Periodic Family of Balls

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Tiivistelmä – Referat – Abstract

Maisterintutkielmassani "Spectral Gaps in the Dirichlet Problem for the Laplacian on a Cylinder in R3 Perforated by a Periodic Family of Balls" todistetaan, että tietyillä ehdoilla Laplace-operaattorin essentaalinen spektri sisältää mielivaltaisen ennalta-annetun määrän "aukkoja". Tarkemmin sanottuna tutkielma käsittelee Dicrichlet'n ongelmaa sylinterissä, josta on poistettu jaksollinen perhe tarpeeksi suuria kuulia.Tämän todistuksen avainkohdat ovat oikean jaksollisuus-solun määrittelemisessä, Friedrichsin epäyhtälön todistamisessa kyseiselle geometrialle, sekä essentiaalisen spektrin aukkojen ylä- ja alarajojen löytämisessä. Suurin osa käytetystä matemaattisesta koneistosta nojaa funktionaalianalyysiin, ja tarkemmin Sobolev-avaruuksien teoriaan, joka käsittelee osittaisdifferentiaaliyhtälöiden niin sanottuja heikkoja ratkaisuja. Käytämme ongelman ratkaisemisessa myös aikaisemman tutkimuksen ideoita, kuten Floquet-Bloch-muunnosta, jonka ansiosta ongelma voidaan koko sylinterin sijaan ratkaista äärellisessä jaksollisuus-solussa.

Avainsanat – Nyckelord – Keywords

Laplace Operator, Spectral Gaps, Perforated Domain

Säilytyspaikka – Förvaringställe – Where deposited

Muita tietoja – Övriga uppgifter – Additional information

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Chapter 1

Introduction

1.1 Abstract

In this thesis we prove that the essential spectrum σ_{ess} of the Laplace operator $-\Delta$ with respect to the Dirichlet problem in a cylinder $C \subset \mathbb{R}^3$ perforated by a singly-periodic family of sufficiently large balls contains any a priori number of gaps. The result is obtained from asymptotic analysis of the cell spectral problem. It can be considered a generalization of earlier and similar results that hold in $\mathbb{R}^2$, to a special case in $\mathbb{R}^3$.

1.2 Motivation and previous results

The idea for the topic of this thesis came from previous research conducted by S.A. Nazarov, K. Ruotsalainen and J. Taskinen in their paper *Spectral Gaps in the Dirichlet and Neumann Problems on the Plane Perforated by a Double-Periodic Family of Circular Holes* [6], and that of F. Ferraresso and J. Taskinen in their paper *Singular Perturbation Dirichlet Problem in a Double-Periodic Perforated Plane* [3]. The main results presented in this work are already known for holes of general shape $\{|x_1|^\mu + |x_2|^\mu \leq r\}$, $1 < \mu < \infty$

perforating $\mathbb{R}^2$, and in particular, the result holds for circular holes, i.e. when $\mu = 2$. We continue in the spirit of these earlier works by examining a similar problem in a domain in $\mathbb{R}^3$. Using techniques developed in the forementioned papers, while developing some techniques of our own, we prove a special case where a 3-dimensional cylinder is perforated by a singly-periodic family of balls.

The main differences between earlier works and our new considerations lie in the geometry and the dimension of the domain. The properties of the domain are crucial as they define key properties, such as boundedness of the Laplacian $-\Delta$. In addition to the mathematical point of view, some of the interest for this topic arises from a more physical interpretation of the problem, as certain geometric situations admit gaps in the spectrum of operators appearing in physics. Earlier works [3], [6] motivate the study of the spectral problem in $\mathbb{R}^2$ with certain physical examples modelled by the propagation of surface waves over a layer of an ideal fluid with a double-periodic family of obstacles, i.e. the linear water wave equation. Similarly, we are motivated by physical phenomena in $\mathbb{R}^3$, namely, propagation of longitudinal waves. It is known that a wave's propagation in a given medium is affected by the presence of gaps in the spectrum of the differential operator by making the waves unable to propagate in the frequency corresponding to the spectral gaps.

In our 3-dimensional case it is natural to consider 3-dimensional physical settings and phenomena, such as longitudinal waves propagating through a medium. Common examples of longitudinal waves are acoustic waves, i.e. sounds moving through air, or pressure waves in water. Another interesting example arises from a special case of electromagnetic waves called plasma waves. Electromagnetic waves are usually transverse waves, yet plasma waves, which exist in plasma and confined spaces, are of longitudinal nature. Another application area could be band gap engineering, a process where controlling the band gap allows one to create desirable electrical properties in materials. One can also imagine that the results could see use in the field of acoustics, where preventing certain vibrations from propagating is desirable.

Chapter 2

Preliminaries

In this chapter we will walk the reader through a brief exposition of prerequisite knowledge that is necessary for the understanding of the context and the results of this thesis. Most of the material covered consists of definitions and theorems of *functional analysis*, a branch of mathematical analysis dealing with vector spaces that have some additional structure (topology, norm, inner product), and the linear functions on these spaces. In addition to the brief walkthrough provided here, it is recommended that the reader has some background in this topic, for example by attending an introductory course in functional analysis. This chapter will only cater the bare minimum amount of information in order to serve readability, and thus proofs of the prerequisite theorems are omitted. In addition to this, some definitions of the basic concepts, e.g. what is a metric space or an inner product, are omitted. The proofs and standard definitions presented here can be found in most standard textbooks on this topic. Although we aim to be brief in our exposition, we have included enlightening remarks throughout this chapter, which are intended to provide further motivation and insight to the topic.

2.1 Prerequisite definitions and theorems

Definition 2.1.1. [**Banach space**] A normed vector space X over the scalar field $\mathbb{R}$ or $\mathbb{K}$ equipped with a norm $\|\cdot\|_\Omega$ so that that X is complete with respect to the norm is called a *Banach space*. By complete with respect to the norm we mean that every Cauchy sequence (x_n) in X converges to an element in X, i.e. there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Definition 2.1.2. [**Dual space**] Given a normed vector space V over a field $\mathbb{K}$, the *dual space* V^* is the set of all bounded linear mappings $\phi : V \to \mathbb{K}$. These mappings in V^* are also called *linear functionals*. A linear functional ϕ is called *bounded* if $\exists M \in \mathbb{K}$ s.t. $M \geq 0$ and $\forall x \in H$ holds $\|\phi(x)\| \leq |M| \, \|x\|$.

Next we recall the definition of a Hilbert space.

Definition 2.1.3. [**Hilbert space**] A real or complex vector space H is called an *inner product space* if it is equipped with an *inner product* $(\cdot,\cdot)_H$, for which it holds for all $x, y, x_1, x_2 \in H$ and $a, b \in \mathbb{C}$ that

(1) $(x, y)_H = \overline{(y, x)_H}$.

(2) $(ax_1 + bx_2, y)_H = a(x_1, y)_H + b(x_2, y)_H$

(3) $(x, x)_H \geq 0$, where (c) is an equality $\Leftrightarrow x = 0$.

An inner product space H that is a complete metric space with respect to the distance function induced by the inner product $(\cdot,\cdot)_H$ is called a *Hilbert space*.

Definition 2.1.4. [**Sesquilinear form**] Let H be a complex vector space. A mapping $B : H \times H \to \mathbb{C}$ is a called a *sesquilinear form* if for all $x, y, z \in H$ and $a, b \in \mathbb{C}$ the following holds

(1) $B(x+y,z) = B(x,z) + B(y,z)$,

(2) $B(x,y+z) = B(x,y) + B(x,z)$,

(3) $B(ax,by) = a\bar{b}B(x,y)$.

Definition 2.1.5. [**L^p spaces**] The space of functions for which the p-th power of the absolute value is Lebesgue integrable is denoted $L^p(\Omega)$, where Ω is a domain in $\mathbb{R}^n$. For a given $p \in [1,\infty[$, functions in $L^p(\Omega)$ that agree almost everywhere are identified with each other, making them equivalence classes of functions. Usually, however, they are still referred to as functions and used in similar fashion. The space $L^p(\Omega)$ is endowed with the norm

$$\|f\|_p = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}, \tag{2.1.1}$$

making it a normed vector space over $\mathbb{R}$ or $\mathbb{K}$. For $f \in L^p(\Omega)$ it holds that the L^p-norm is finite:

$$\|f\|_p = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} < \infty. \tag{2.1.2}$$

The only exponent making the space $L^p(\Omega)$ a Hilbert space is $p = 2$ and we call this the *space of square integrable functions* $L^2(\Omega)$.

Definition 2.1.6. [**Weak derivative**] A *weak derivative* is a generalization of derivative for functions u that are not necessarily differentiable, but which are Lebesque integrable, i.e. $u \in L^1(\Omega)$, where $\Omega \subset \mathbb{R}$. Let $I \subset \mathbb{R}$ be an open interval. We then say that $v \in L^1(I)$ is a *weak derivative* of u if it holds that

$$\int_I u(t)\varphi'(t)dt = -\int_I v(t)\varphi(t)dt, \quad \forall \varphi \in C_0^\infty(I), \tag{2.1.3}$$

where $t \in I \subset \mathbb{R}$ and $C_0^\infty(I)$ is the space of smooth functions with compact support.

For $x \in \Omega \subset \mathbb{R}^n$ and a multi-index α (which is an n-tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ of non-negative integers), the *α-th weak partial derivative* of u is defined as

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\varphi(x) dx, \quad \forall \varphi \in C_0^\infty(\Omega), \tag{2.1.4}$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$, and the α-th weak partial derivative is denoted $D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = v$.

Remark 2.1.7. With $k \in \mathbb{N}$, $u \in C^k(\Omega)$ and $|\alpha| \leq k$ the definitions of the weak partial derivative and the classical partial derivative coincide. Here, $C^k(\Omega)$ is the set of all k-times continuously differentiable functions. Also, if the weak derivative v of u exists, it is unique up to a set of Lebesque measure zero.

Definition 2.1.8. **[Sobolev spaces $W_p^l(\Omega)$]** We denote the *Sobolev space* of functions by $W_p^l(\Omega)$, where $1 \leq p < \infty$, $l \in \mathbb{N}$, and Ω is an open subset of $\mathbb{R}^n$. We say that a function $u \in L^p(\Omega)$ is in the Sobolev space $W_p^l(\Omega)$ if the there exists weak derivatives $D^\alpha u$ for all $|\alpha| \leq l$ such that

$$\partial^\alpha u \in L^p(\Omega), \ |\alpha| \leq l. \tag{2.1.5}$$

We can then write $u \in W_p^l(\Omega)$. The Sobolev spaces $W_p^l(\Omega)$ are equipped with the norm

$$\|u\|_{W_p^l(\Omega)} = \left(\sum_{|\alpha| \leq l} \int_\Omega |\partial^\alpha u|^p dx \right)^{\frac{1}{p}}. \tag{2.1.6}$$

Throughout the thesis, we only consider the Sobolev space $W_2^1(\Omega)$, that is, $p = 2$, $l = 1$. Sobolev spaces with $p = 2$ are Hilbert spaces and it is customary to denote $W_2^1(\Omega) =$

$H^1(\Omega)$. All Sobolev spaces are complete with respect to their standard norm and therefore are Banach spaces. The importance of Sobolev spaces comes from the the fact that the functions in these spaces have sufficiently many derivatives for the study of differential equations. They are thought to be the natural space for the study of differential equations as the solutions are found in these spaces instead of the spaces of continuous functions C^k, where the derivatives are interpreted in the classical way. In this work, the Sobolev space of interest is $H_0^1(\Omega)$, which is defined as the completion of the space $C_0^\infty(\Omega)$ of compactly supported test functions with respect to the norm of the Sobolev space $H^1(\Omega)$. Here the subscript $_0$ denotes that the functions vanish on the boundary $\partial\Omega$, we present the Sobolev norm in this space for the convenience of the reader:

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega \|\nabla u\|^p dx + \int_\Omega |u|^p dx \right)^{\frac{1}{p}}, \tag{2.1.7}$$

.

Definition 2.1.9. **[Lipschitz domain]** Let Ω be an open subset of $\mathbb{R}^n$ and let $\partial\Omega$ denote the boundary of Ω. Then Ω is called a *Lipschitz domain* if for every point $p \in \partial\Omega$ there exists a hyperplane P of dimension $n-1$, i.e. an $n-1$-dimensional vector subspace of $\mathbb{R}^n$, containing p, a Lipschitz-continuous function $g : P \to \mathbb{R}^n$ on the hyperplane, and $r > 0$ and $h > 0$ such that

$$\Omega \cap C = \{\, x + y\bar{n} \mid x \in B_r(p) \cap P,\ -h < y < g(x) \,\}, \tag{2.1.8}$$

$$\partial\Omega \cap C = \{\, x + y\bar{n} \mid x \in B_r(p) \cap P,\ g(x) = y \,\}, \tag{2.1.9}$$

where $\bar{n}$ is a unit vector that is normal to the hyperplane P,

$B_r(p) := \{ \ x \in \mathbb{R}^n \mid \|x - p\| < r \ \}$ and $C := \{ \ x + y\bar{n} \mid x \in B_r(p) \cap P, \ -h < y < h \ \}$.

This means that the boundary $\partial\Omega$ can be seen locally as the graph of a Lipschitz continuous function, i.e. it is sufficiently regular.

Definition 2.1.10. [**Gauss-Green Formula**] For $u, v \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^n$, the Gauss-Green formula states that

$$\int_\Omega \Delta u \cdot v dx = \int_{\partial\Omega} (n, \nabla u)_{C^1} \cdot v dS - \int_\Omega (\nabla u) \cdot \nabla v dx, \tag{2.1.10}$$

where $n \in \mathbb{R}^n$ is the unit normal of Ω, dS indicates the Lebesque measure along the $n - 1$-dimensional surface.

A homogeneous Dirichlet boundary value problem can have *classical* and *weak* solutions. We're going to elaborate on the difference through an example boundary value problem

$$\begin{cases} -\Delta u = f \ , \ \ u \in \Omega \\ u = 0 \ , \ \ u \in \partial\Omega. \end{cases} \tag{2.1.11}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Assuming $f \in C(\Omega)$, a *classical* solution of the boundary value problem is a function $u \in C^2(\Omega)$ and such that u is continuous on the boundary and satisfies the boundary conditions pointwise. Often proving the existence of classical solutions is difficult and introducing so-called weak solutions allows one to find solutions to a broader class of problems. By multiplying the equation $-\Delta u = f$ by a smooth function $v \in C_0^\infty(\Omega)$ and using Gauss-Green formula yields

$$-\int_\Omega \Delta u v dx = \int_\Omega (\nabla u) \cdot \nabla v dx = \int_\Omega (\nabla u, \nabla v)_{C_0^\infty} dx = \int_\Omega f v dx, \tag{2.1.12}$$

where the first term on the right hand side of the Gauss-Green formula vanishes due to $u = 0$ on the boundary $\partial\Omega$. We note that $C_0^\infty(\Omega)$ is a dense subset of the Sobolev space $H_0^1(\Omega)$ and that we can pick the test function $v \in H_0^1(\Omega)$, making the last equality in (2.11) hold with weaker assumptions. With this consideration, the *weak solutions* to the boundary value problem (2.11) are $u \in H_0^1(\Omega)$ such that

$$\int_\Omega (\nabla u, \nabla v)_{H_0^1} dx = \int_\Omega f v dx \tag{2.1.13}$$

holds for all $v \in H_0^1(\Omega)$.

Remark 2.1.11. Classical solutions of a boundary value problem are also weak solutions, but the converse doesn't hold without additional assumptions.

Definition 2.1.12. **[Linear operators in Hilbert Spaces]** A linear operator $A : V \to W$ between two normed spaces V and W is called a *bounded linear operator* when the following holds:

$$\exists M \geq 0 \; s.t. \; \forall v \in V \;\; : \|Av\|_W \leq |M| \, \|v\|_V \, , \tag{2.1.14}$$

where $M \in \mathbb{R}$. A linear operator between normed spaces is bounded if and only if it is continuous. Linear operators can also be *unbounded*: consider general, not necessarily bounded, linear operators from Hilbert space H to H as a mapping $A\colon D(A) \subset H \to H$, where $D(A)$ is a linear subspace of H that satisfies the following linearity relation:

$$A(ax + bx) = aAx + bAy \tag{2.1.15}$$

for all $a, b \in \mathbb{C}$, and $x, y \in D(A)$. The space $D(A)$ is called the *domain* of A, and we say that A is *densely defined* if the closure satisfies $\overline{D(A)} = H$.

Definition 2.1.13. [**Trace operator**] For a bounded $\Omega \subset \mathbb{R}^n$ with a boundary that is Lipschitz, the trace Tu of a function $u \in W_p^1(\Omega)$ is defined via a bounded linear operator called the *trace operator* $T : W_p^1(\Omega) \to L^p(\partial\Omega)$ for which it holds that

$$
\begin{aligned}
&Tu = u|_{\partial\Omega}, && u \in W_p^1(\Omega) \cap C(\overline{\Omega}) \\
&\|Tu\|_{L^p(\partial\Omega)} \leq c(p, \Omega)\|u\|_{W_p^1(\Omega)}, && u \in W_p^1(\Omega),
\end{aligned}
\tag{2.1.16}
$$

where $c(p, \Omega)$ is a constant depending on the domain Ω and p. The definition of the trace operator extends the usual idea of restricting a function to the boundary of its domain to functions in Sobolev space $W_p^1(\Omega)$.

Definition 2.1.14. [**Compact operator**] A linear operator $T : X \to Y$ between two Banach spaces is X and Y such that the image of any bounded subset of X has a compact closure, or is *relatively compact*, in Y is called a *compact operator*. An equivalent formulation is that the closure of the image of the unit ball of X under T is relatively compact in Y. A compact operator is also a bounded operator and in particular continuous.

Definition 2.1.15. [**Self-adjoint operator**] A bounded linear operator $A : H \to H$, where H is a Hilbert space, is called *self-adjoint* if for its adjoint operator A^* holds $A = A^*$. For all $x, y \in H$, the adjoint operator is defined through

$$
(Ax, y)_H = (x, A^*y)_H \tag{2.1.17}
$$

Self-adjointness then becomes

$$
(Ax, y)_H = (x, Ay)_H. \tag{2.1.18}
$$

For unbounded operators, the self-adjointness is sensitive to the choice of domain.

Definition 2.1.16. **[Positive operator]** A bounded linear operator $P : H \to H$, where H is a Hilbert space, is called a *positive operator* if $P = A^2$ for some self-adjoint operator $A : H \to H$. The self-adjoint operator A is also called the *positive square root* of P. Equivalently, P is called a positive operator if for all $x \in H$ holds $(Px, x)_H \geq 0$.

Definition 2.1.17. **[Laplace operator]** The *Laplacian* or the *Laplace operator* Δ is a second order differential operator in $\mathbb{R}^n$ (or $\mathbb{K}^n$) given by

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}, \qquad x = (x_1, ..., x_n) \in \mathbb{R}^n \tag{2.1.19}$$

$$\Delta : C^k(\mathbb{R}^n) \to C^{k-2}(\mathbb{R}^n). \tag{2.1.20}$$

Remark 2.1.18. For any domain $\Omega \subset R^n$, the Laplace operator is a self-adjoint unbounded operator from the domain

$$D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega). \tag{2.1.21}$$

The negative Laplacian $-\Delta$ is positive and for all $u \in H_0^1(\Omega)$ we can write:

$$(-\Delta u, u)_{L^2(\Omega)} = (\nabla u, \nabla u)_{L^2(\Omega)} \geq 0. \tag{2.1.22}$$

Theorem 2.1.19. **[Lax-Milgram]** Let H be a Hilbert space and $a(\cdot,\cdot)$ a bilinear form on H s.t. $a(\cdot,\cdot)$ is

$$\textit{bounded}: \ |a(u,v)| \leq C \left\|u\right\| \left\|v\right\| \quad \textit{and} \tag{2.1.23}$$

$$\textit{coercive}: \ a(u,u) \geq c \left\|u\right\|^2 \tag{2.1.24}$$

for some constants $C, c > 0$. Then for any $f \in H^*$, there exists a unique solution $u \in H$ to the equation $a(u,v) = f(v)$ and $\|u\| \leq \frac{1}{c}\|f\|_{H^*}$. In essence, Lax-Milgram is a statement that the continuous linear functionals f in the Hilbert space H and the continuous bilinear forms $a(\cdot,\cdot)$ are in 1-to-1 correspondence and given $f \in H$ we can find a unique $u \in H$ that is a solution to our bilinear form $a(\cdot,\cdot)$. As an immediate consequence, coercive boundary value problem defined via f can be studied using the properties of the bilinear form $a(\cdot,\cdot)$. As Hilbert spaces are antilinearly isomorphic to their *dual space* H^* through *Riesz representation theorem* we can view an element f of the said Hilbert space H as a linear functional on H. In order to make this more tangible to the reader, we present the following defintions and theorems.

Theorem 2.1.20. **[Riesz representation]** Let $h^* \in H^*$ be a bounded linear functional on a Hilbert space H, then there exists $h \in H$ for all $f \in H$, such that

$$h^*(f) = (f,h)_H. \tag{2.1.25}$$

Moreover, $\|h^*\|_{H^*} = \|h\|_H$.

Now as the dual H^* of a Hilbert space H is also a Hilbert space and H^* is anti-isomorphic to H through Riesz representation: $H \cong H^*$. Thus H can be isometrically identified with its dual H^*. As an example of the 1-to-1 correspondence between continuous linear functionals f on a Hilbert space H and continuous bilinear forms $a(\cdot,\cdot)$, we set $H = H_0^1(\Omega)$ and look at a bilinear form defined as

$$a(\cdot,\cdot) : H \times H \to \mathbb{R}, \tag{2.1.26}$$

$$a(u,v) = \int_\Omega (\nabla u, \nabla v)_H dx\ ,\ u,v \in H. \tag{2.1.27}$$

We then define a linear functional f on $H_0^1(\Omega)$ as

$$f(v) = \int_\Omega fvdx \ , \ v \in H. \tag{2.1.28}$$

We then arrive to the *weak formulation* of the problem, which is to find $u \in H$ such that for all $v \in H$ the following equality holds:

$$a(u, v) = f(v). \tag{2.1.29}$$

Interpreted this way, the weak formulation lets us convert certain differential equations into an equality of linear functionals on the Hilbert space H. As weak formulation makes no claim about the boundary conditions, they need to be included into the definition of the underlying function space, e.g. functions in the Sobolev space $H_0^1(\Omega)$ need to vanish on the boundary $\partial\Omega$, and this is essentially the same as having Dirichlet boundary conditions for the related differential equation.

Definition 2.1.21. **[Spectrum of an operator]** Let $A\colon D(A) \to H$ be a densely defined, not necessarily bounded operator. The *resolvent set* of A is defined as

(2.1.30)

$$\varrho(A) := \{\lambda \in \mathbb{C}\colon \text{the operator } R_\lambda := (A - \lambda I)^{-1} \text{ exist as bounded operator from } H \text{ onto } D(A)\}.$$

The *spectrum* $\sigma = \sigma(A)$ of the operator A is defined as the complement

$$\sigma(A) = \mathbb{C} \setminus \varrho(A). \tag{2.1.31}$$

Definition 2.1.22. **[Essential spectrum of an operator]** Let σ be the spectrum of a self-adjoint linear operator A. The closed subset $\sigma_{\text{ess}} \subseteq \sigma$, called the *essential spectrum*, contains the values $\lambda \in \sigma$ for which, for the operator $A - \lambda I$, the kernel $N(A - \lambda I) := \{x \in D(A - \lambda I)\colon (A - \lambda I)x = 0\}$ or the cokernel $H \setminus \operatorname{Im}(A - \lambda I)$ is infinite-dimensional. Furthermore, the complement of the essential spectrum, $\sigma_{\text{disc}} = \sigma \setminus \sigma_{\text{ess}}$ is called the discrete spectrum, which in the scope of this thesis, coincides with the normal eigenvalues of the self-adjoint linear operator A. The essential spectrum σ_{ess} of a self-adjoint operator A is closed and contained in $\mathbb{R}$.

Chapter 3

Geometric Setting

In this work, the domain of the Dirichlet problem for the Laplacian $-\Delta$ is the cylinder $C \subset \mathbb{R}^3$ perforated by a singly-periodic family of holes, defined as follows:

$$\Pi_R := C \setminus \bigcup_{\alpha \in 2\mathbb{Z}+1} \overline{B_R(\alpha)}. \tag{3.0.1}$$

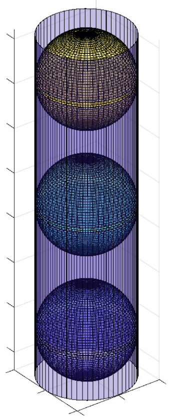

Figure 3.0.1: The domain $\Pi_R := C \setminus \bigcup_{\alpha \in 2\mathbb{Z}+1} \overline{B_R(\alpha)}$ with $R = \frac{1}{2}$, or alternatively $\varepsilon = 0$.

The cylinder $C \subset \mathbb{R}^3$ itself is defined as

$$C := \{x \in \mathbb{R}^3 \mid \sqrt{x_1^2 + x_2^2} - \frac{1}{2} \leq 0\}. \tag{3.0.2}$$

The usual definition of balls of radius $\frac{1}{2} \geq R > 0$ centered at the origin is

$$B_R := \{x \in \mathbb{R}^3 \mid |x| < R, R \in (0, \frac{1}{2}]\}. \tag{3.0.3}$$

If adjacent balls share a point, it is not clear if a complicated singularity in our boundary value problem would occur. In this work, we prevent such situation from happening by defining singly-periodic family of balls of radius $R > 0$ as

$$B_R(\alpha) := \{x \in \mathbb{R}^3 \mid (x_1, x_2, x_3 - \alpha) \in B_R,\ \alpha \in 2\mathbb{Z} + 1\}. \tag{3.0.4}$$

From now on, we set $R = \frac{1}{2} - \varepsilon$.

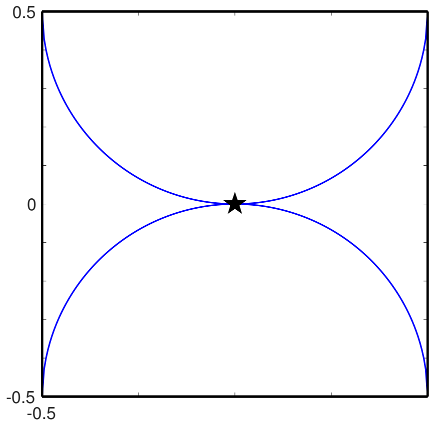

Figure 3.0.2: If we choose the spacing of the balls in the cylinder differently, for example $\alpha \in \mathbb{Z} + \frac{1}{2}$, adjacent balls can share a point. When $R = \frac{1}{2}$, the adjacent balls touch at the origin (marked with the black star), which can cause more complicated singularities to occur.

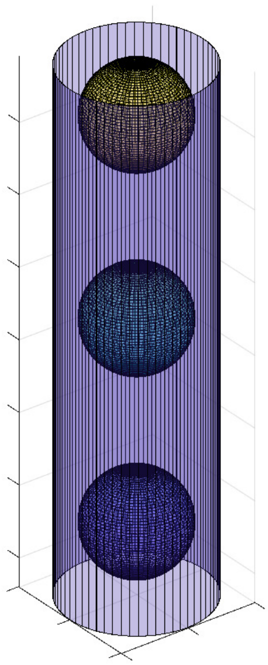

Figure 3.0.3: The domain $\Pi_R := C \setminus \bigcup_{\alpha\in 2\mathbb{Z}+1} \overline{B_R(\alpha)}$ with some fixed $\varepsilon > 0$.

We define the periodicity cell as

$$
\begin{aligned}
Q_\varepsilon := \{x \in \mathbb{R}^3 \mid\ & \left(|x_1|^2 + |x_2|^2 + |x_3 - \alpha_j|^2\right)^{\frac{1}{2}} > \frac{1}{2} - \varepsilon,\ j = 0, 1\ , \\
& \left(|x_1|^2 + |x_2|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}, |x_3| < 1\},
\end{aligned} \tag{3.0.5}
$$

where $\varepsilon \in [0, \frac{1}{2}]$ and $\alpha_j = (-1)^j$.

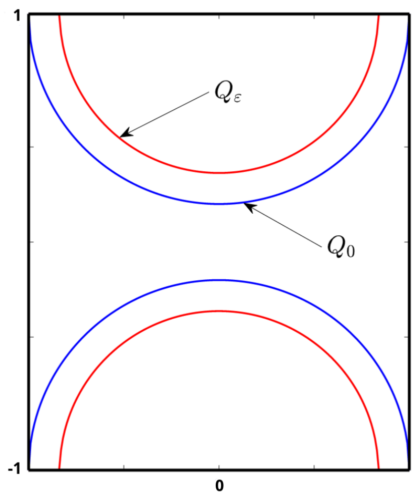

Figure 3.0.4: 2-dimensional cross-section of the periodicity cell Q_ε with $t = 0$, $\varepsilon = 0$, or alternatively $R = \frac{1}{2}$ (in blue), and $\varepsilon = 0.1$ (in red).

Moreover, the annuli in the periodicity cell are defined as

$$A_\varepsilon^{\alpha_j} := \{x \in \overline{Q_\varepsilon} \mid x_3 = \alpha_j \ , \ \varepsilon \in [0, \frac{1}{2}] \ \}. \tag{3.0.6}$$

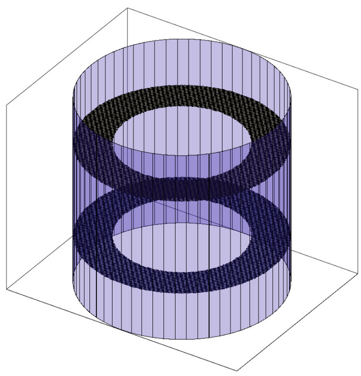

Figure 3.0.5: The annuli $A_\varepsilon^{\alpha_j}$ are central to many problems in this work due to the quasi-periodicity conditions arising from the Floquet-Bloch transform.

Chapter 4

Dirichlet problem and the Floquet-Bloch transform

The spectral Laplace problem with homogenous Dirichlet conditions is of the following form:

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Pi_R, \\ u(x) = 0, & x \in \partial\Pi_R, \end{cases} \tag{4.0.1}$$

where Δ is the Laplace operator with respect to $x \in \Pi_R$, and $\lambda \in \mathbb{R}$ is the spectral parameter which is contained on the real axis due to self-adjointness of the Laplace operator Δ. As defined in the previous section, Π_R is the cylinder $C \subset \mathbb{R}^3$ perforated by singly periodic family of balls $B_R(\alpha)$.

In order to confine the consideration of the infinitely tall cylinder into a bounded domain, we apply the *Floquet-Bloch transform* to turn our geometric setting into a finite periodicity cell. In this work, we only sketch the Floquet-Bloch theory, which gives a connection between the essential spectrum σ_{ess} of the boundary value problem (4.0.1) and the discrete spectrum of a family of model problems. We will concentrate on the properties of the solutions of these problems. The details on Floquet-Bloch theory can be found in [4].

In the geometric setting of this thesis, the Floquet-Bloch transform for functions u on Π_R, is defined as

$$u \mapsto U(x,\eta) = \frac{1}{2\pi} \sum_{\alpha \in 2\mathbb{Z}+1} e^{-i\eta(x+[0,0,\alpha])} u(x+[0,0,\alpha]), \tag{4.0.2}$$

where $x \in Q_\varepsilon$ and $\eta \in (-\pi, \pi]$.

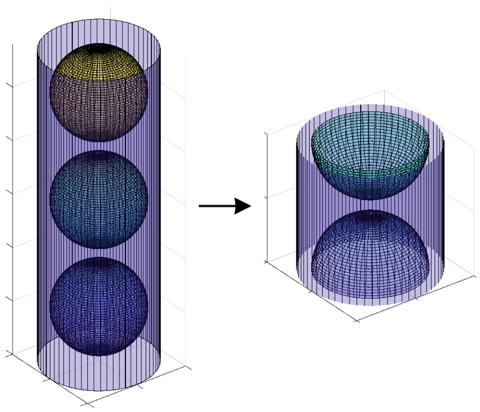

Figure 4.0.1: Floquet-Bloch transform takes the problem from the cylinder to the periodicity cell.

Since the periodicity will only happen in one direction due to the geometry, we have the following parameter $\eta \in (-\pi, \pi]$-dependent problem in the periodicity cell Q_ε:

$$
(4.0.3)\qquad \begin{cases} -\Delta U(x,\eta) = \Lambda^\varepsilon(\eta) U(x,\eta), & x \in Q_\varepsilon, \\ U(x,\eta) = 0, & x \in \partial Q_\varepsilon \setminus \overline{A_0^{-1} \cup A_0^1}, \\ U\big|_{A_\varepsilon^{-1}} = e^{i\eta} U\big|_{A_\varepsilon^1}, \end{cases}
$$

where $\Lambda^\varepsilon(\eta)$ is a new notation for the spectral parameter λ. The subspace of $H^1(Q_\varepsilon)$ satisfying the Dirichlet boundary and η-dependent quasi-periodicity conditions of the problems (4.0.3) is denoted as

$$
(4.0.4)\qquad \mathcal{H}^\varepsilon(\eta) := \{V \in H^1(Q_\varepsilon) \mid V(x) = 0 \;\forall x \in \partial Q_\varepsilon \setminus \overline{A_\varepsilon^{-1} \cup A_\varepsilon^1}\;,\; V\big|_{A_\varepsilon^{-1}} = e^{i\eta} V\big|_{A_\varepsilon^1}\}.
$$

$\mathcal{H}^\varepsilon(\eta)$ is a closed subspace of $H^1(Q_\varepsilon)$, and it is itself also a Hilbert space because it is closed by the topology induced by the inner product of $H^1(Q_\varepsilon)$.

Next, we consider the weak solutions of problem (4.0.3), which are the solutions $U \in \mathcal{H}^\varepsilon(\eta)$ of the variational identity

$$
(4.0.5)\qquad (\nabla U, \nabla V)_{Q_\varepsilon} = \Lambda_n^\varepsilon (U, V)_{Q_\varepsilon}, \quad \forall V \in \mathcal{H}^\varepsilon(\eta),
$$

where ∇ is the gradient operator, and $(\cdot,\cdot)_{Q_\varepsilon}$ is the inner product in the Lebesgue space $L^2(Q_\varepsilon)$. Based on the results given in chapter (5), it can be shown that the variational problem (4.0.5) can be associated with a self-adjoint operator $\mathcal{A}$ with domain $\mathcal{H}^\varepsilon(\eta)$. Furthermore, due to the compactness of the embedding of $\mathcal{H}^\varepsilon(\eta)$ into $L^2(Q_\varepsilon)$, the spectrum of the operator $\mathcal{A}$ is discrete, which means that for a fixed η the spectrum of the problem is given by the increasing sequence of eigenvalues

$$
(4.0.6)\qquad 0 \le \Lambda_1^\varepsilon(\eta) \le \Lambda_2^\varepsilon(\eta) \le \Lambda_3^\varepsilon(\eta) \le \cdots \to \infty
$$

where multiplicities have been taken into account. We note that we are also making the well-justified assumption that the eigenfunctions $(U_n^\varepsilon(\eta))_{n\in\mathbb{N}}$ form a complete orthonormal set in $L^2(Q_\varepsilon)$. Moreover, from [3] we know that the functions

$$\eta \mapsto \Lambda_n^\varepsilon(\eta) \tag{4.0.7}$$

are continuous. From this it follows that the spectral segments

$$\Upsilon_n = \{\Lambda_n^\varepsilon(\eta) \mid \eta \in (-\pi, \pi]\} \tag{4.0.8}$$

are continuous, closed real intervals, which may be overlapping. The Floquet theory further states that the essential spectrum σ_{ess} of the original problem (4.0.1), and the spectra of problems (4.0.3) are related by

$$\sigma_{\text{ess}} = \bigcup_{n\in\mathbb{N}} \Upsilon_n. \tag{4.0.9}$$

The main goal of this work is to find the upper and lower estimates for the endpoints of each Υ_n.

Later in this work, we want to consider the eigenfunctions and eigenvalues of the same variational problem in Q_0, where $\varepsilon = 0$. This will be of interest since we want to find a way to get upper and lower limits for the spectral gaps of 4.0.5. It is worth noticing that singularities might occur at the annuli, and therefore the domain is no longer guaranteed to be Lipschitz. Setting $\varepsilon = 0$, we arrive at the limit case problem

$$\begin{cases} -\Delta U(x) = \Lambda^0 U(x,\eta), & x \in Q_0, \\ U(x,\eta) = 0, & x \in \partial Q_0 \end{cases} \tag{4.0.10}$$

and the corresponding subspace of the Sobolev space $H^1(Q_0)$ satisfying

$$\mathcal{H}^0 := \{V \in H^1(Q_0) \mid V(x) = 0 \ \forall x \in \partial Q_0\}. \tag{4.0.11}$$

The variational formulation of the limit problem in Q_0 is given by

$$(\nabla U, \nabla V)_{Q_0} = \Lambda_n^0 (U, V)_{Q_0}, \quad \forall V \in \mathcal{H}^0, \tag{4.0.12}$$

where $U \in \mathcal{H}^0$. Later, we want to show that the positive self-adjoint operator $\mathcal{A}^0$ associated with the left hand side of then variational form 4.0.12 has a discrete spectrum. To do this, it will be sufficient to show that the embedding $\iota : \mathcal{H}^0 \hookrightarrow L^2(Q_0)$ is compact. This will require a separate treatment of the neighborhoods of the annuli, since it is not necessarily clear that the eigenfunctions are bounded in this domain.

We note that the η-dependent periodicity conditions vanish as $\varepsilon \to 0$. Therefore, if we can verify the inequality

$$|\Lambda_n^\varepsilon(\eta) - \Lambda_n^0| \leq \mathbf{c}_n \varepsilon^\gamma, \quad \varepsilon \in (0, \epsilon_n], \quad n \in \mathbb{N}, \tag{4.0.13}$$

where $\gamma > 0$, $\mathbf{c}_n > 0$, we know that the spectral segments Υ_n are within $\mathbf{c}_n \varepsilon^\gamma$ neighborhoods of the eigenvalues Λ_n^0 of the limit case problem. By considering the eigenvalues with multiplicity μ, that is,

$$\Lambda_{n-1}^0 < \Lambda_n^0 = \cdots = \Lambda_{n+\mu-1}^0 < \Lambda_{n+\mu}^0, \tag{4.0.14}$$

we can see that the parameter $\mathbf{c}_n \varepsilon^\gamma$ can be made smaller than the distance between two non-equal eigenvalues by choosing a sufficiently small ε. We denote by U_n^0 the eigenfunction corresponding to Λ_n and assume the normalization

$$(U_n^0, U_m^0)_{L^2(Q_0)} = \delta_{m,n}. \tag{4.0.15}$$

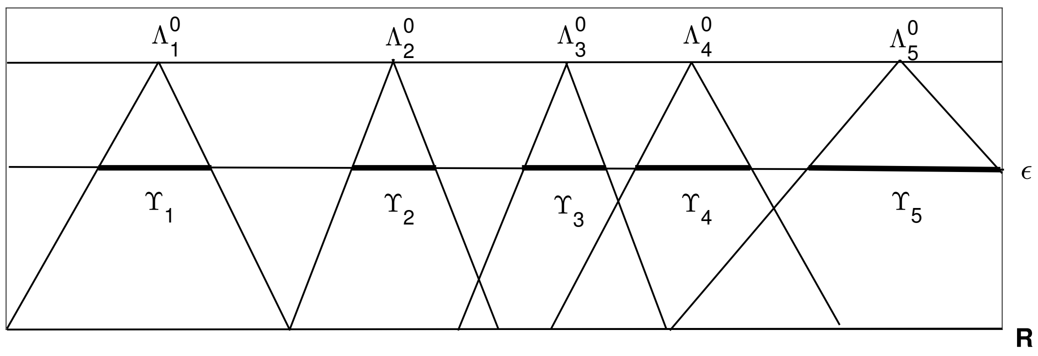

Figure 4.0.2: Illustration of the relationship between discrete spectrum eigenvalues Λ_n^0 and the spectral segments Υ_n.

Chapter 5

Compact embedding and the Discrete spectrum

In this chapter we introduce a result which states that the spectrum of a positive, self-adjoint operator associated with our problem has a discrete spectrum, provided that the embedding of our Sobolev space $H^1(Q_0)$ into $L^2(Q_0)$ is compact. We give the *Rellich-Kondrachov compactness theorem*, which gives the concrete conditions when a Sobolev space $W^{1,p}(\Omega)$ can be compactly embedded into the space of q-integrable functions $L^q(\Omega)$. We also present relevant results from *Spectral theory of self-adjoint operators in Hilbert space* by M.S. Birman and M.Z. Solomjak [1].

Theorem 5.0.1 (Rellich-Kondrachov compactness theorem). *Let $\Omega \in \mathbb{R}^n$ be a bounded, open Lipschitz domain. Suppose $1 \leq p < n$. Then the embedding*

$$\iota : W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \tag{5.0.1}$$

is compact for all $q \in [1, p^*)$, $p^* := \frac{np}{n-p}$. In our case, $p = q = 2$ and the inclusion becomes

$$\iota : H^1(\Omega) \hookrightarrow L^2(\Omega). \tag{5.0.2}$$

Proof: [2], page 272.

Theorem 5.0.2. *Given a positive self-adjoint operator $A\colon D(A) \to H$ that can be written as a product $A = A^{\frac{1}{2}}A^{\frac{1}{2}}$, where $A^{\frac{1}{2}}$ is a of positive self-adjoint operator, there exists a unique closed sesquilinear form associated with A. The form a is defined by*

$$a[x, y] = (A^{\frac{1}{2}}x, A^{\frac{1}{2}}y)_{D(a)}, \ x, y \in D(a) = D(A^{\frac{1}{2}}). \tag{5.0.3}$$

On the other hand, every closed, positive sesquilinear form is associated with a unique positive self-adjoint operator.

Proof: [1], Chapter 10: Theorems 1 and 2.

Theorem 5.0.3. *The spectrum of a positive self-adjoint operator A is discrete if and only if the embedding operator ι is a compact operator from $D(a)$ to H.*

Proof: [1], Chapter 10: Theorem 5.

The previous theorems are given in the original notation from Birman & Solomjak. We emphasize that in our case 5.0.2 becomes:

$$a[f, u] = (\nabla f, \nabla u)_{D(A)}, \quad f, u \in D(a) = \mathcal{H}^0. \tag{5.0.4}$$

Here A is negation of the Laplace operator $-\Delta$, which is a positive self-adjoint operator, $D(a) = \mathcal{H}^0$, and $H = L^2(Q_0)$. In the next section, we prove the compactness of our embedding $\iota : \mathcal{H}^0 \hookrightarrow L^2(Q_0)$ using Rellich-Kondrachov compactness theorem and a classical result of functional analysis, *Friedrich's inequality*, a generalization of the *Poincaré-Wirtinger inequality*.

Theorem 5.0.4 (Friedrich's inequality). *Let Ω be a bounded subset of $\mathbb{R}^n$ with diameter d. For a function $U : \Omega \to \mathbb{R}$, $U \in W_p^l(\Omega)$ and for which the trace is zero on the boundary $\partial\Omega$, we have:*

$$\frac{1}{d^l}\|U\|_{L^p(\Omega)} \leq \left(\sum_{|\alpha|=l}\|D^\alpha U\|^p_{L^p(\Omega)}\right)^{1/p}. \tag{5.0.5}$$

5.1 Compactness of the inclusion

To assess the compactness of the inclusion $\iota : \mathcal{H}^0 \hookrightarrow L^2(Q_0)$, we first define for $\delta > 0$ the partitioning of the periodicity cell Q_0 into sets $Q_0(\delta)$ and $R_0(\delta)$, given by

$$Q_0(\delta) := \{x \in Q_0 \mid |x_3| < 1 - \delta\}, \tag{5.1.1}$$

$$R_0(\delta) := Q_0 \setminus \overline{Q_0(\delta)}. \tag{5.1.2}$$

Theorem 5.1.1. *For all $\delta > 0$ the inclusion $\iota : \mathcal{H}^0 \hookrightarrow L^2(Q_0)$ can be represented as the sum of compact $K\colon H_0^1(Q(\delta)) \to L^2(Q_0)$ and bounded $J : H^1(R_0(\delta)) \to L^2(R_0(\delta))$ with an operator norm estimate of $||J||_{L^2} = \mathcal{O}(\delta^2)$. In particular, ι is a compact embedding.*

Proof: With a fixed $x_3 \in [\frac{1}{2}, 1)$, we notice that if $U \in \mathcal{H}^0$, then $U(x_3, \cdot)$ lies in the space $H_0^1(A(x_3))$, where

$$A(x_3) := \{(x_1, x_2) \in \mathbb{R}^2 \mid \Theta(x_3) < (x_1^2 + x_2^2)^{\frac{1}{2}} < \frac{1}{2}\} \tag{5.1.3}$$

is the annulus defined at a fixed x_3 and

$$\Theta(x_3) = |(\frac{1}{2})^2 - (1 - x_3)^2|^{\frac{1}{2}} \text{ when } x_3 \in [\frac{1}{2}, 1), \text{ and } 0 \text{ otherwise.} \tag{5.1.4}$$

is the x_3-dependent chord length of the sphere perforating the cylinder. This comes from the chord length formula $l = 2\sqrt{r_c^2 - d_c^2}$, where l is the length of a chord, r_c is the radius of the circle and d_c is the perpendicular distance from the chord to the center of the circle. To a fixed x_3 we denote the polar coodinates of a point $(x_1, x_2) \in A(x_3)$ by $r \in [\Theta(x_3), \frac{1}{2}]$ and $\phi \in [0, 2\pi]$. The area of the annulus $A(x_3)$ at x_3 is obtained by integrating the radial component r from $\Theta(x_3)$ to $\frac{1}{2}$, and further integrating this over central angle ϕ from 0 to 2π. With this notion, and further recalling that the periodicity conditions are not necessary in $H_0^1(A(x_3))$, we use the one-dimensional Friedrichs' inequality to get

$$\frac{1}{(\frac{1}{2} - \Theta(x_3))^2} \int_{\Theta(x_3)}^{\frac{1}{2}} |U(r, \phi, x_3)|^2 dr \leq \int_{\Theta(x_3)}^{\frac{1}{2}} |\partial_r U(r, \phi, x_3)|^2 dr, \tag{5.1.5}$$

where $\frac{1}{2} - \Theta(x_3)$ corresponds to the length of the radial component of annulus $A(x_3)$.

We introduce a smooth positive function ρ on the set $\mathcal{Q} := Q_0 \setminus \overline{A_0^{-1} \cup A_0^1}$ such that ρ coincides with with $\operatorname{dist}(x, A_0^{-1})$ and $\operatorname{dist}(x, A_0^1)$ for all x in the neighborhoods of A_0^{-1} and A_0^1 in Q_0 respectively. This is motivated by the notion that as $\operatorname{dist}(x, A_0^\alpha)$ gets very small, the quantity $\frac{1}{\rho^2}$ gets arbitrarily large. When investigating the behavior of the eigenfunctions U near A_0^α, we want to find out if they tend to zero in any arbitrarily small neighborhood of annuli A_0^α. Dividing U by a power of the distance function and taking the L^2-norm will show us, if finite, that U has superpolynomial decay near the annuli, and therefore the extension by zero to a larger Q_ε done later in this work will be justified.

First, we note that for $x_3 \geq \frac{1}{2}$ it holds that:

$$
(5.1.6)\qquad \begin{aligned}
|1-x_3| &= (1-x_3)\\
&= \tfrac{1}{2}-(x_3-\tfrac{1}{2})\\
&= \tfrac{1}{2}-((x_3-\tfrac{1}{2})^2)^{\frac{1}{2}}\\
&= \tfrac{1}{2}-(x_3^2-x_3+(\tfrac{1}{2})^2)^{\frac{1}{2}},
\end{aligned}
$$

and

$$
(5.1.7)\qquad \begin{aligned}
\left(\tfrac{1}{2}-\Theta(x_3)\right) &= \tfrac{1}{2}-|(\tfrac{1}{2})^2-(1-x_3)^2|^{\frac{1}{2}}\\
&= \tfrac{1}{2}-|(\tfrac{1}{2})^2-x_3^2+2x_3-1|^{\frac{1}{2}}.
\end{aligned}
$$

Comparing the radicands of the above expressions, we get:

$$
(5.1.8)\qquad \begin{aligned}
f(x) := x_3^2-x_3+(\tfrac{1}{2})^2 \quad &\text{vs.} \quad g(x) := (\tfrac{1}{2})^2-x_3^2+2x_3-1 \qquad ||-x_3^2+x_3-\tfrac{1}{4}\\
0 \quad &\text{vs.} \quad -2x_3^2+3x_3-1
\end{aligned}
$$

We know that $h(x) := -2x_3^2+3x_3-1$ is a positive function on the interval $[\frac{1}{2},1)$. It follows that $g(x)\geq f(x)$ on the interval $[\frac{1}{2},1)$. Therefore,

$$
(5.1.9)\qquad |1-x_3| \geq \left(\frac{1}{2}-\Theta(x_3)\right).
$$

Finally, we consider such x_3 that $\rho(x)$ coincides with the distance function $|1-x_3|$. We integrate both sides of inequality 5.1.5 with respect to ϕ and x_3 to get

$$
(5.1.10)\qquad \begin{aligned}
\left\|\frac{U}{\rho^2}\right\|^2_{L^2(Q_0^+)} &= \left\|\frac{U}{|\frac{1}{2}-x_3|^2}\right\|^2_{L^2(Q_0^+)}\\
&\underset{5.1.9}{\leq} \left\|\frac{U}{\left(\frac{1}{2}-\Theta(x_3)\right)^2}\right\|^2_{L^2(Q_0^+)}\\
&= \int_0^1\int_0^{2\pi}\frac{1}{\left(\frac{1}{2}-\Theta(x_3)\right)^2}\int_{\Theta(x_3)}^{\frac{1}{2}}|U(r,\phi,x_3)|^2 dr d\phi dx_3\\
&\underset{5.0.4}{\leq} c\int_0^1\int_0^{2\pi}\int_{\Theta(x_3)}^{\frac{1}{2}}|\partial_r U(r,\phi,x_3)|^2 dr d\phi dx_3 \leq c\|\nabla U\|^2_{L^2(Q_0^+)}.
\end{aligned}
$$

Here, Q_0^+ is the upper part of Q_0 where $x_3 \geq 0$ and the first inequality follows from (5.1.9), and the second is due to Friedrichs' inequality (5.0.4). We get the analoguous result for $x_3 \leq 0$ due to symmetry, and hence the result is valid in the entire Q_0. We will use this result to prove the boundedness of the inclusion below.

Next we set $\delta > 0$, and define a smooth non-negative cut-off function $\chi_\delta \in C^\infty(\mathbb{R}^3)$ such that

$$\chi_\delta(x) := \begin{cases} 1, & \text{if } \operatorname{dist}(x, A_0^{-1} \cup A_0^1) \leq \delta, \\ 0, & \text{if } \operatorname{dist}(x, A_0^{-1} \cup A_0^1) > 2\delta. \end{cases} \tag{5.1.11}$$

We can now write any $U \in H^1(Q_0)$ as

$$U = \chi_\delta U + (1 - \chi_\delta)U \equiv \chi_\delta U + g, \tag{5.1.12}$$

where $g \in H^1(Q_0)$ has a support in $Q_0(\delta)$, which is an open Lipschitz domain.

By Rellich-Kondrachov theorem (5.0.1) we know that the embedding of $H^1(Q_0(\delta))$ into $L^2(Q_0)$ is compact. It now remains to be proven that $\chi_\delta U$ has a L^2-norm of order $\mathcal{O}(\delta^2)$:

$$\begin{aligned} \|\chi_\delta U\|_{L^2}^2 = &\int_{Q_0} |\chi_\delta U|^2 dx \leq \int_{R(2\delta)} \rho^4 |\rho^{-2} U|^2 dx \\ \leq &(2\delta)^4 \int_{R(2\delta)} |\rho^{-2} U|^2 dx \underset{5.1.10}{\leq} c_1 \delta^4 \|U\|_{H^1(R(2\delta))}^2 \leq c_2 \delta^4. \end{aligned} \tag{5.1.13}$$

Here the last two inequalities follow from the previous notion that $\|\rho^{-2}U\|_{L^2}^2 \leq c\|\nabla U\|_{L^2}^2 \leq c(\|U\|_{L^2} + \|\nabla U\|_{L^2})^2$, which is the Sobolev norm in $H^1(R(2\delta))$. □

From the compactness of previous embedding it follows that the self-adjoint operator associated with the limit problem has a discrete spectrum (5.0.3). We denote the unbounded sequence of eigenvalues and corresponding eigenvectors with

$$(\Lambda_n^0)_{n\in\mathbb{N}} \quad , \quad (U_n^0)_{n\in\mathbb{N}} \tag{5.1.14}$$

and emphasize that the eigenfunctions are orthonormal, i.e. $(U_m, U_n)_{Q_0} = \delta_{m,n}$.

Chapter 6

Behavior of eigenfunctions U at the annuli A_0^α

The domains of boundary value problems for elliptic equations are said to be *singularly perturbed* if they have perturbations that can cause singularities to occur. In contrast, the perturbations of the domain are said to be *regular* when the boundary of the domain is a closed, smooth surface. Examples of singular features in domains are small holes, edges, round corners, small slits and thin ligaments.

The boundary value problem in this work is considered in the domain Q_ε, which depends on a small parameter ε, where the boundary of the domain Q_0 is not smooth and contains two singular rings, A_0^{-1} and A_0^1. These are the limit cases of the annuli as $\varepsilon \to 0$. As we transition from Q_0 to Q_ε, these rings convert into small annuli, A_ε^{-1} and A_ε^1. We are concerned with the fact that as ε tends to zero, the sharp corner formed in between the cylinder and the perforating ball could cause the domain Q_0 to not be sufficiently smooth. Smoothness is a requirement for extending our domain by zero, as the Sobolev embedding $H^1(Q_0) \hookrightarrow H^1(\overline{Q_0})$ could otherwise fail. Furthermore, by local estimates for solutions of elliptic boundary value problems [5], the eigenfunctions are smooth everywhere in the periodicity cell Q_0, except maybe in A_0^{-1}, A_0^1, and hence this area deserves a closer inspection.

In this chapter we show, that perturbing our periodicity cell by letting $\varepsilon \to 0$, our eigenfunctions U tend to zero *exponentially* fast, in particular, the decay to zero is then *super-polynomial.* The infinite differentiability is preserved at the discontinuity rings, and the eigenfunctions U belong to $C^\infty(\overline{Q_0})$. The domain Q_0 is therefore a sufficiently smooth for the extension by zero used later in this work.

Definition 6.0.1 (Extension by zero). *For $U \in H^1(Q_0)$ we may define an extension by zero to $H^1(\overline{Q_\varepsilon})$ by*

$$\widetilde{U}(x) = \begin{cases} U(x), & x \in Q_0 \\ 0, & otherwise. \end{cases} \tag{6.0.1}$$

6.1 Exponential decay of eigenfunctions U

Let ε and δ be such that $0 < \varepsilon < 2\varepsilon < \delta < 1$, and let $r(x) = \operatorname{dist}(x, A_0^{-1} \cup A_0^1)$. Define a regularized distance function $\mathcal{D}(x)$ as

$$\mathcal{D}(x) = \begin{cases} e^{\beta/\varepsilon}, & \text{if } r(x) < \varepsilon, \\ e^{\beta/r(x)}, & \text{if } \varepsilon \le r(x) < \delta, \\ e^{\beta/\delta}, & \text{if } r(x) \ge \delta, \end{cases} \tag{6.1.1}$$

for all $x \in Q_0$, $\beta > 0$. If U is an eigenfunction of limit problem 4.0.12, then also $\mathcal{D}^2 U \in \mathcal{H}^0$. We denote $\mathcal{V} = \mathcal{D}U$ and get the following:

(6.1.2)

$$\begin{aligned}
\Lambda_0 \int_{Q_0} \mathcal{D}^2 U^2 dx \underset{4.0.10}{=} & \int_{Q_0} (\nabla U) \cdot \nabla(\mathcal{D}^2 U) dx = \int_{Q_0} (\nabla U) \cdot \nabla(\mathcal{D}\mathcal{V}) dx \\
= & \int_{Q_0} \mathcal{D}(\nabla U) \cdot \nabla \mathcal{V} dx + \int_{Q_0} \mathcal{V}(\nabla U) \cdot \nabla \mathcal{D} dx \\
= & \int_{Q_0} (\nabla(\mathcal{D}U) - U(\nabla \mathcal{D})) \cdot \nabla \mathcal{V} dx + \int_{Q_0} (\nabla(\mathcal{V}U) - U(\nabla \mathcal{V})) \cdot \nabla \mathcal{D} dx \\
= & \int_{Q_0} \nabla \mathcal{V} \cdot \nabla \mathcal{V} dx - \int_{Q_0} U(\nabla \mathcal{D}) \cdot \nabla \mathcal{V} dx + \int_{Q_0} U(\nabla \mathcal{V}) \cdot \nabla \mathcal{D} dx \\
& + \int_{Q_0} \mathcal{V}(\nabla U) \cdot \nabla \mathcal{D} dx - \int_{Q_0} U(\nabla \mathcal{V}) \cdot \nabla \mathcal{D} dx \\
= & \int_{Q_0} |\nabla \mathcal{V}|^2 dx - \int_{Q_0} U(\nabla \mathcal{D}) \cdot \nabla(\mathcal{D}U) dx + \int_{Q_0} (\mathcal{D}U)(\nabla U) \cdot \nabla \mathcal{D} dx \\
= & \int_{Q_0} |\nabla \mathcal{V}|^2 dx - \int_{Q_0} U(\nabla \mathcal{D}) \cdot (\mathcal{D}(\nabla U) + U(\nabla \mathcal{D})) dx + \int_{Q_0} (\mathcal{D}U)(\nabla U) \cdot \nabla \mathcal{D} dx \\
= & \int_{Q_0} |\nabla \mathcal{V}|^2 dx - \int_{Q_0} U(\nabla \mathcal{D}) \cdot \mathcal{D}(\nabla U) dx + \int_{Q_0} (\mathcal{D}U)(\nabla U) \cdot \nabla \mathcal{D} dx - \int_{Q_0} |\nabla \mathcal{D}|^2 U^2 dx \\
= & \int_{Q_0} |\nabla \mathcal{V}|^2 dx - \int_{Q_0} |\nabla \mathcal{D}|^2 U^2 dx.
\end{aligned}$$

This gives us

$$\int_{Q_0} |\nabla \mathcal{V}|^2 dx = \int_{Q_0} |\nabla \mathcal{D}|^2 \frac{\mathcal{V}^2}{\mathcal{D}^2} dx + \Lambda_0 \int_{Q_0} \mathcal{D}^2 U^2 dx. \tag{6.1.3}$$

We note that for all $x \in Q_0$ it holds that

$$\begin{gathered}
|\nabla \mathcal{D}(x)|^2 \leq \beta^2 r(x)^{-4} \mathcal{D}(x)^2, \\
e^{\beta/\delta} \leq \mathcal{D}(x) \leq e^{\beta/\varepsilon}.
\end{gathered} \tag{6.1.4}$$

Using these inequalities we get

$$\int_{Q_0} |\nabla \mathcal{V}|^2 dx \leq \beta^2 \int_{Q_0} \frac{\mathcal{V}^2}{r^4} dx + \Lambda_0 e^{2\beta/\varepsilon} \int_{Q_0} U^2 dx. \tag{6.1.5}$$

By the inequality 5.1.10 and the boundedness of $r(x)$ we get, for some constant $c > 0$ independent of ε, that

$$(c - \beta^2) \int_{Q_0} \frac{\mathcal{V}^2}{r^4} dx \leq \Lambda_0 e^{2\beta/\varepsilon} \int_{Q_0} U^2 dx. \tag{6.1.6}$$

We now choose a β such that $0 < \beta^2 < c$. This leads to

$$e^{-\frac{2\beta}{\varepsilon}} \int_{Q_0} \frac{\mathcal{V}^2}{r^4} dx \leq c_\lambda < \infty. \tag{6.1.7}$$

For integration, define the sets

$$\begin{aligned} \mathcal{I}_1 &:= \{x \in Q_0 \mid r(x) \leq e^{-\beta/2\varepsilon}\}, \\ \mathcal{I}_2 &:= \{x \in Q_0 \mid r(x) > e^{-\beta/2\varepsilon}\}. \end{aligned} \tag{6.1.8}$$

From inequality 6.1.7 we get that both of the integrals

$$e^{-\frac{2\beta}{\varepsilon}} \int_{Q_0 \cap \mathcal{I}_1} \frac{\mathcal{V}^2}{r^4} dx, \quad e^{-\frac{2\beta}{\varepsilon}} \int_{Q_0 \cap \mathcal{I}_2} \frac{\mathcal{V}^2}{r^4} dx \tag{6.1.9}$$

are bounded for all $\varepsilon > 0$. Using the first one, we get

$$\int_{Q_0 \cap \mathcal{I}_1} \mathcal{V}^2 dx \leq e^{-\frac{2\beta}{\varepsilon}} \int_{Q_0 \cap \mathcal{I}_1} \frac{\mathcal{V}^2}{r^4} dx < \infty \tag{6.1.10}$$

for all $\epsilon > 0$, since for all $x \in \mathcal{I}_1$ it holds that

$$r(x) \leq e^{-\beta/2\varepsilon} \Leftrightarrow 1 \leq e^{-\beta/2\varepsilon} r^{-1} \Leftrightarrow 1 \leq e^{-2\beta/\varepsilon} r^{-4}. \tag{6.1.11}$$

This implies that both $\mathcal{V}$, and in particular the eigenfunctions U, have exponential decay in L^2-norm in a neighborhood of the annuli, thus point-wise at $A_0^{-1} \cup A_0^1$. Therefore the eigenfunctions are in $C^\infty(\overline{Q_0})$, and can be extended by zero to a larger domain Q_ε.

Chapter 7

Estimates for eigenvalues $\Lambda_n^\varepsilon(\eta)$

In this chapter, we obtain the lower and upper bound estimates for the eigenvalues $\Lambda_n^\varepsilon(\eta)$ for the family of model problems introduced in (4.0.3). These eigenvalues $\Lambda_n^\varepsilon(\eta)$ will then be used to construct spectral segments Υ_n, which are in a direct connection with the original Dirichlet boundary value problem for the Laplacian (4.0.1) through (4.0.9). Both of these estimates will be obtained from the eigenvalues Λ_n^0. These estimates are enough to demonstrate and prove the existence of spectral gaps between the segments Υ_n, given a sufficiently small ε. This shows that we can create gaps in the essential spectrum σ_{ess} of the original problem by manipulating ε.

7.1 Upper bound estimate

We start by finding an upper bound estimate for eigenvalues $\Lambda_n^\varepsilon(\eta)$. The eigenvalues of the ε-perturbated problem are given by the min-max formula (found in [2]):

$$\Lambda_n^\varepsilon(\eta) = \max_{\mathcal{H}_n^\varepsilon(\eta)} \inf_{U} \frac{(\nabla U, \nabla U)_{Q_\varepsilon}}{(U, U)_{Q_\varepsilon}}, \tag{7.1.1}$$

where the infimum is taken over non-zero functions $U \in \mathcal{H}_n^\varepsilon(\eta)$, where $\mathcal{H}_n^\varepsilon(\eta)$ is an arbitrary linear subspace of $\mathcal{H}^\varepsilon$ with a co-dimension $n-1$.

Next, we want to use the functions from Q_0 in the inner product to obtain an estimate that depends on the discrete eigenvalues Λ_n^0 of the limit problem (4.0.10). To achieve this, we use the results of the previous section to justify the extension by zero of the eigenfunctions U^0 in Q_0 to the larger domain Q_ε. For convenience, we use the same notation for the extended eigenfunctions in Q_ε as for the ones in Q_0. Since the original set is a complete orthonormal set, the extensions are linearly independent. From this, it follows that each subspace $\mathcal{H}_n^\varepsilon(\eta)$ contains a non-trivial linear combination

$$\mathbf{U} = \sum_{k=1}^{n} a_k U_k^0, \quad \sum_{k=1}^{n} |a_k| = 1, \tag{7.1.2}$$

where a_k are dependend on the subspace $\mathcal{H}_n^\varepsilon(\eta)$.

We note that the eigenvalues $\Lambda_1^0, \dots, \Lambda_n^0$ associated with the eigenfunctions $U_1^0, \dots, U_n^0$ are taken in an increasing order, and $(U_m^0, U_n^0) = \delta_{m,n}$ due to orthonormality. From this and by substituting $\mathbf{U}$ into the min-max formula, it follows that

$$\begin{aligned} \Lambda_n^\varepsilon(\eta) &\leq \max_{\mathcal{H}_n^\varepsilon(\eta)} \frac{(\nabla \mathbf{U}, \nabla \mathbf{U})_{Q_\varepsilon}}{(\mathbf{U}, \mathbf{U})_{Q_\varepsilon}} = \max_{\mathcal{H}_n^\varepsilon(\eta)} \frac{\sum_{k=1}^n a_k \overline{a_k} (\nabla U_k^0, \nabla U_k^0)_{Q_\varepsilon}}{\sum_{k=1}^n a_k \overline{a_k} (U_k^0, U_k^0)_{Q_\varepsilon}} \\ &= \max_{\mathcal{H}_n^\varepsilon(\eta)} \frac{\sum_{k=1}^n |a_k|^2 \Lambda_k^0}{\sum_{k=1}^n |a_k|^2} \leq \Lambda_n^0, \end{aligned} \tag{7.1.3}$$

where we used the first condition of the Dirichlet problem, that is $\Delta U_k^0 = \Lambda_k^0 U_k^0$. This gives us the upper bound $\Lambda_n^\varepsilon(\eta) \leq \Lambda_n^0$. $\square$

7.2 Lower bound estimate

To obtain a lower bound estimate for $\Lambda_n^\varepsilon(\eta)$, we want to use similar reasoning as in the previous section, and to apply techniques and arguments developed in [6]. We want to use the eigenfunctions in Q_ε and scale the domain to fit inside Q_0. We then extend the functions by zero to the rest of Q_0. This is motivated by the fact that after appropriately definining the scaling of the periodicity cell Q_ε, and a cut-off function $X_{\sqrt{\varepsilon}}(x)$, we can use the results obtained in [6] as is, with the difference that the cut-off only needs to happen in x_3-direction. We again fix $n \in \mathbb{N}$, and start with the min-max formula

$$\Lambda_n^0 = \max_{\mathcal{H}_n^0} \inf_U \frac{(\nabla U, \nabla U)_{Q_0}}{(U,U)_{Q_0}}, \tag{7.2.1}$$

where $\mathcal{H}_n^0$ is an arbitrary linear subspace of $\mathcal{H}^0$ with co-dimension $n-1$.

Because of the quasiperiodicity conditions in the annuli in Q_ε, the eigenfunctions are not required to vanish on the border of the periodicity cell, and hence cannot be extended by zero. To get around this, we consider a scaled down version of Q_ε. This is accomplished by defining, for a given $\varepsilon > 0$, a mapping

$$\phi_\varepsilon : (x_1, x_2, x_3) \mapsto (x_1, x_2, (1 - 2\sqrt{\varepsilon})x_3). \tag{7.2.2}$$

Now, the domain $\widehat{Q}_\varepsilon = \phi_\varepsilon(Q_\varepsilon)$ is completely contained in Q_0 for all $\varepsilon \in [0, \frac{1}{2}[$. Namely, if $x \in Q_\varepsilon$, we have for all $\phi_\varepsilon(x)$:

$$\begin{aligned} & x_1^2 + x_2^2 + |(1-2\sqrt{\varepsilon})x_3|^2 \\ = \quad & x_1^2 + x_2^2 + x_3^2 + 4\varepsilon + 1 - 2x_3 > \frac{1}{2}, \end{aligned} \tag{7.2.3}$$

since $x_1^2 + x_2^2 + x_3^2 - 2x_3 + 1 > \frac{1}{2} - \varepsilon$ for all $x \in Q_\varepsilon$. Therefore, $\phi_\varepsilon(x) \in Q_0$. It is apparent that $\operatorname{dist}(\widehat{Q}_\varepsilon, \partial Q_0) \leq 2\sqrt{\varepsilon}$. We define a smooth cut-off function $\chi_\varepsilon(x)$ such that

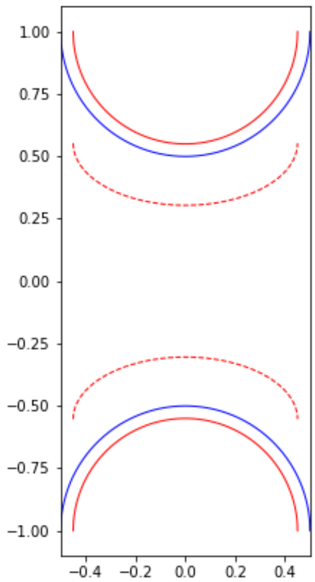

Figure 7.2.1: The cross sections of the scaled domain Q_0 (blue), Q_ε (red), and $\widehat{Q}_\varepsilon$ (dashed).

$$\chi_\varepsilon(x) = \begin{cases} 0 & \operatorname{dist}(x, \partial Q_0) \leq 4\sqrt{\varepsilon} \\ 1 & \operatorname{dist}(x, \partial Q_0) \geq 5\sqrt{\varepsilon} \end{cases} \tag{7.2.4}$$

and such that for all $x \in \widehat{Q}_\varepsilon$ holds $|\nabla \chi_\varepsilon(x)| \leq \frac{c}{\sqrt{\varepsilon}}$. We note that $\chi_\varepsilon(x) = 0$ for all $x \in \partial\widehat{Q}_\varepsilon$. It follows from these definitions that the function

$$\widehat{Q}_\varepsilon \ni x \mapsto \mathcal{U}_k^\varepsilon(x) := \chi_\varepsilon(x) U_k^\varepsilon(\phi_\varepsilon^{-1}(x)) \tag{7.2.5}$$

can be extended by zero to $\mathcal{H}^0$. The motivation behind this definition is to have a $\sqrt{\varepsilon}$ amount of space between the cutoff points. This allows us to use the tricks and formulas developed in [6]. In particular, because of lemma 3.1. in [6], the eigenfunctions of problem

(4.0.3) in the domain $\widehat{Q}_\varepsilon$ satisfy the inequalities

$$\|(\rho+\varepsilon^{\frac{1}{4}})^{-2}\nabla U_n^\varepsilon\| + \|(\rho+\varepsilon^{\frac{1}{4}})^{-4}U_n^\varepsilon\| \le C_n, \tag{7.2.6}$$

where ρ is the weight in the estimate from Friedrichs' inequality (5.1.10) and the majorant C_n depending on $n\in\mathbb{N}$ of the eigenfunction can be taken to be the same for all $\varepsilon\in(0,\varepsilon_n]$ with some $\varepsilon_n>0$.

Next we verify that the functions $\mathcal{U}_1^\varepsilon,\dots,\mathcal{U}_n^\varepsilon$ are linearly independent. This is equivalent to any subspace $\mathcal{H}_n^0$ in the min-max formula (7.2.1) containing the linear combination

$$\mathbf{U}^\varepsilon = \sum_{k=1}^n a_k^\varepsilon \mathcal{U}_k^\varepsilon, \quad \sum_{k=1}^n |a_k^\varepsilon| = 1. \tag{7.2.7}$$

By orthogonality and normalization we get

$$\begin{aligned}(\mathcal{U}_k^\varepsilon,\mathcal{U}_m^\varepsilon)_{Q_0} &= (1+2\sqrt{\varepsilon})^{-2}(\chi_\varepsilon U_k^\varepsilon\circ\phi_\varepsilon^{-1},\chi_\varepsilon U_m^\varepsilon\circ\phi_\varepsilon^{-1})_{Q_\varepsilon}\\ &= (1+2\sqrt{\varepsilon})^{-2}\delta_{k,m}+\mathcal{O}(\sqrt{\varepsilon})\end{aligned} \tag{7.2.8}$$

since χ_ε is a function that differs from 1 only in a $c\sqrt{\varepsilon}$-neighborhood of the annuli $A_\varepsilon^{-1}, A_\varepsilon^1$. We have

$$\begin{aligned}|((1-\chi_\varepsilon)U_k^\varepsilon,(1-\chi_\varepsilon)U_m^\varepsilon)_{Q_\varepsilon}| &\le \|U_k^\varepsilon\|_{Q_\varepsilon^*}\|U_m^\varepsilon\|_{Q_\varepsilon^*}\\ &\le c\varepsilon^2\|(\rho+\sqrt{\varepsilon})^{-2}U_k^\varepsilon\|_{Q_\varepsilon^*}\|(\rho+\sqrt{\varepsilon})^{-2}U_m^\varepsilon\|_{Q_\varepsilon^*}\\ &\le c\varepsilon^2\|\nabla U_k^\varepsilon\|_{Q_\varepsilon^*}\|\nabla U_m^\varepsilon\|_{Q_\varepsilon^*}\\ &\le c\varepsilon^2\Lambda_k^\varepsilon(\eta)^{1/2}\Lambda_m^\varepsilon(\eta)^{1/2}\le C_n\varepsilon^2,\end{aligned} \tag{7.2.9}$$

where

$$Q_\varepsilon^* := \{x\in Q_\varepsilon \mid \ \rho(x)<c\sqrt{\varepsilon}\} \tag{7.2.10}$$

is the union of $c\sqrt{\varepsilon}$-neighborhoods of the annuli. We return to the min-max formula

$$\frac{\|\nabla U^\varepsilon\|_{Q_\varepsilon}^2}{\|U^\varepsilon\|_{Q_\varepsilon}^2} \tag{7.2.11}$$

with the inner product in the numerator given explicitly by

$$(\nabla \mathcal{U}_k^\varepsilon, \nabla \mathcal{U}_n^\varepsilon) = (\chi_\varepsilon \nabla U_k^\varepsilon + U_k^\varepsilon \nabla \chi_\varepsilon \ , \ \chi_\varepsilon \nabla U_m^\varepsilon + U_m^\varepsilon \nabla \chi_\varepsilon)_{Q_\varepsilon}. \tag{7.2.12}$$

In order to find the lower bound estimate, we need the relations

$$\begin{aligned}
&(\nabla U_k^\varepsilon, \nabla U_m^\varepsilon)_{Q_\varepsilon} = \Lambda_k^\varepsilon(\eta)\delta_{k,m},\\
&\|\chi_\varepsilon \nabla U_k^\varepsilon\|_{Q_\varepsilon} \le \Lambda_k^\varepsilon(\eta)^{\frac{1}{2}} \le C_n^{\frac{1}{2}},\\
&\|U_k^\varepsilon \nabla \chi_\varepsilon\|_{Q_\varepsilon} \le c\varepsilon^{-\frac{1}{2}}\|U_k^\varepsilon\|_{Q_\varepsilon^*} \le c\varepsilon^{\frac{1}{2}}\|(\rho+\varepsilon^{\frac{1}{4}})^{-4}U_k^\varepsilon\| \le C_n\varepsilon^{\frac{1}{2}},\\
&\|(1-\chi_\varepsilon)^{\frac{1}{2}}\nabla U_k^\varepsilon\|_{Q_\varepsilon} \le \|\nabla U_k^\varepsilon\|_{Q_\varepsilon^*} \le c\varepsilon^{\frac{1}{2}}\|(\rho+\varepsilon^{\frac{1}{4}})^{-2}\nabla U_k^\varepsilon\|_{Q_\varepsilon^*} \le C_n\varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{7.2.13}$$

The verification of these relations can be done by using the same arguments as in (7.2.9) and by applying inequality (7.2.6). By using test functions $\mathbf{U}^\varepsilon \in \mathcal{H}_n^0$ from (7.2.7), we can now derive from the min-max principle the inequality

$$\Lambda_n^0 \le \sup_{\mathcal{H}_n^0} \frac{(\nabla \mathbf{U}^\varepsilon, \nabla \mathbf{U}^\varepsilon)_{Q_\varepsilon}}{(\mathbf{U}^\varepsilon, \mathbf{U}^\varepsilon)_{Q_\varepsilon}}. \tag{7.2.14}$$

Following the calculations and tricks introduced in [6], we can estimate the right-hand side as follows:

$$\sup_{\mathcal{H}_n^0} \frac{(\nabla \mathbf{U}^\varepsilon, \nabla \mathbf{U}^\varepsilon)_{Q_\varepsilon}}{(\mathbf{U}^\varepsilon, \mathbf{U}^\varepsilon)_{Q_\varepsilon}} \le \frac{\sum_{k=1}^n |a_k^\varepsilon|^2 \Lambda_k^\varepsilon(\eta) + c_n\varepsilon^{\frac{1}{2}}}{(1+2\sqrt{\varepsilon})^{-2}\left(\sum_{k=1}^n |a_k^\varepsilon|^2 - c_n\varepsilon\right)} \le (1+C_n\sqrt{\varepsilon})^2\Lambda_n^\varepsilon(\eta). \tag{7.2.15}$$

Finally, combining the upper and lower bound estimates obtained above, we can conclude our results as a theorem.

Theorem 7.2.1. *Given any $n \in \mathbb{N}$, there exist numbers $\epsilon_n > 0$ and $\mathbf{c}_n > 0$ such that for any $\varepsilon \in (0, \epsilon_n]$ and $\eta \in (-\pi, \pi]$ the eigenvalues $\Lambda_n^\varepsilon(\eta)$ of the Floquet-Bloch transformed problem (4.0.3) can be estimated as*

$$\frac{1}{(1+2\mathbf{c}_n\sqrt{\varepsilon})^2}\Lambda_n^0 \le \Lambda_n^\varepsilon(\eta) \le \Lambda_n^0, \tag{7.2.16}$$

where Λ_n^0 are the eigenvalues of the 0-cell problem (4.0.10).

Corollary 7.2.2 (Spectral gaps). *Let $n \in \mathbb{N}$, $n \ge 2$, Λ_n^0, and ϵ_n be defined as in Theorem 7.2.1, and a constant $\mathbf{c}_n' \in \mathbb{R}$. Assume also that the eigenvalues Λ_n^0 have a multiplicity μ as in (4.0.14). Then for*

$$\varepsilon \le \min(\epsilon_n, \epsilon_{n-1}) \quad \textit{and} \quad (1-\mathbf{c}_n'\sqrt{\varepsilon})\Lambda_{n-1}^0 \le \Lambda_n^0, \tag{7.2.17}$$

a gap is opened between the spectral segments Υ_{n-1} and Υ_n of the spectrum of the Dirichlet problem (4.0.1). As a direct consequence, given $N \in \mathbb{N}$, the essential spectrum σ_{ess} of the original problem (4.0.1) has at least N gaps, provided that ε is chosen to be sufficiently small.

Chapter 8

Conclusions

In this work, we have extended the previous results from [6], [3] to a unit cylinder perforated by a periodic family of balls in $\mathbb{R}^3$. We showed that the essential spectrum of the Laplace operator for the Dirichlet problem in this domain has any a priori number of gaps. This number depends on the distance between the boundary of the cylinder and the balls within. The main results of this work were obtained by first proving the Friedrichs' inequality in our geometric setting, and together with the help of a new cutoff function and a scaling of the domain specific to our problem, we arrived at reasoning and conclusions similar to what can be found in earlier works, most notably in [6], [3]. We note that the final result depends on the distance between the balls in z-direction, as we need to be able to scale the original domain to the open space between the balls in order to obtain the lower bound estimates for the spectral segments. Interesting futher research on the topic could be done by considering the unit cube as a periodicity cell, and extending the results regarding the existence of spectral gaps to the whole of $\mathbb{R}^3$.

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